Tutorato di Statistica 1 del 27/09/2010
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Esercizio 1.

X v.a., $X \sim N(\mu, \sigma^2)$
$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$
$E[e^{tx}] = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$
$= \frac{e^{t\mu}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{tx} e^{-t\mu} e^{(x-\mu)^2/2\sigma^2} dx =$
$= \frac{e^{t\mu}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-1/2\sigma^2\{(x-\mu)^2 - 2\sigma^2 t(x-\mu) + (\sigma^2 t)^2 - (\sigma^2 t)^2}\} dx =$
$= \frac{e^{t\mu + (\sigma^2 t)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-1/2\sigma^2[(x-\mu)-(\sigma^2 t)]^2} dx =$
$= \frac{e^{t\mu + (\sigma^2 t^2)/2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-z^2/2\sigma^2} dx = e^{t\mu + (\sigma^2 t^2)/2}$
Dunque $m(t) = e^{t\mu + (\sigma^2 t^2)/2}$
$E[X] = m'(t)|_{t=0} = e^{t\mu + (\sigma^2 t^2)/2}(\mu + t\sigma^2)|_{t=0} = \mu$
$Var[X] = E[X^2] - E[X]^2$
$E[X^2] = m^{''}(t)|_{t=0}$
$m^{''}(t) = e^{t\mu + (\sigma^2 t^2)/2}(\mu^2 + t\sigma^2\mu + \sigma^2 + t\mu\sigma^2 + (t\sigma^2)^2)|_{t=0} =$
$= \mu^2 + \sigma^2$
$Var[X] = m^{''}(t)|_{t=0} - m'(t)^2_{t=0} = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Esercizio 2.

X v.a., $X \sim Po(\lambda)$.
$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} 1_{\{0,1,2...\}}(x)$
$m(t) = \sum_{x=0}^{+\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{+\infty} \frac{e^{tx}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{+\infty} \frac{(e^t\lambda)^x}{x!} = e^{-\lambda} e^{e^t\lambda} = e^{\lambda(e^t-1)}$
$E[x] = m'(t)|_{t=0}$
$m'(t) = \lambda e^t e^{\lambda(e^t-1)}|_{t=0} = \lambda$
$Var[X] = E[X^2] - E[X]^2$
$m^{''}(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}|_{t=0} = \lambda + \lambda^2$
$Var[X] = m^{''}(t)|_{t=0} - m'(t)^2_{t=0} = \lambda = \lambda^2 - \lambda^2 = \lambda$

Esercizio 3.
X v.a., $X \sim Unif(a,b)$
$f_X(x) = \frac{1}{b-a}, x \in (a,b)$
$m(t) = \int_a^b e^{tx} \frac{1}{b-a} dx = \frac{1}{t(b-a)} \int_a^b te^{tx} dx = \frac{e^{bt} - e^{at}}{t(b-a)}$
$E[x] = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$
$E[x^2] = \int_a^b x^2 \frac{1}{b-a} = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + a^2 + ab}{3}$
$Var[x] = \frac{b^2 + a^2 + ab}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$

Esercizio 4.

X v.a., $X \sim Exp(\lambda)$
$f_X(x) = \lambda e^{-\lambda x} 1_{[0,+\infty)}(x)$
$m(t) = \int_0^{+\infty} \lambda e^{tx} e^{-\lambda x} dx = \frac{\lambda}{t-\lambda} \int_0^{+\infty} (t-\lambda) e^{-x(\lambda-t)} = \frac{\lambda}{\lambda-t}$, $\lambda > t$
$m'(t) = \frac{\lambda}{(\lambda-t)^2}$
$E[X] = m'(0) = \frac{1}{\lambda}$
$m''(t) = \frac{2\lambda}{(\lambda-t)^3}$
$E[X^2] = m''(0) = \frac{2}{\lambda^2}$
$Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Esercizio 5.

Sia $Y = aX + b$. Sia $M_X(t)$ il momento di X.

$M_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = E[e^{at}e^{tb}] = e^{tb}E[e^{atX}] = e^{tb}M_X(at)$.

Esercizio 6.

Un esperimento consiste nel lancio di due palline in quattro scatole, in modo tale che ogni pallina abbia la stessa probabilità di cadere in una qualsiasi delle scatole. Sia X il numero di palline nella prima scatola, quindi:

$X = \sum_{i=1}^{2} Y_i$ dove $Y_i \sim Bernoulli(p)$, con $p = 1/4$.
Allora $X \sim Binom(2, 1/4)$
quindi si ha:
$P(X = k) = \binom{2}{k}(\frac{1}{4})^k(\frac{3}{4})^{2-k}$, per $k = 0, 1, 2$
$P(X = 0) = \frac{9}{16}$
$P(X = 1) = \frac{6}{16}$
$P(X = 2) = \frac{1}{16}$

$$F_X(x) = \begin{cases} 9/16 & x \leq 0 \\ 15/16 & 0 \leq x \leq 1 \\ 1 & 0 \leq x \leq 2 \end{cases}$$

$E[X] = 2 * 1/4 = 1/2$

$Var(X) = 2 * 1/4 * 3/4 = 3/8$